Colouring of the Surfaces of Three-dimensional Polytopes (The Four-Colour Theorem)

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ABSTRACT

The four-colour map theorem states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour. Two regions are considered to be adjacent if they share a common boundary that is not a corner (a point shared by three or more regions). The theorem was proposed in the 1850s and became the first theorem to be proved by computational methods in the 1970s. Despite the theorem being true, some geopolitical maps require more than four colours (if, for example, some regions are not contiguous) and the theorem has never been of great interest to mapmakers. This paper describes the theorem and explores how it could be extended to three dimensions. We restrict our study to the colouring of the surfaces of three-dimensional polytopes or polyhedra, specifically those that are convex. An analysis of the relationship between two-dimensional maps and three-dimensional surfaces is presented with regard to the minimum number of colours required. Visual examples are provided for regular polyhedral of increasing number of polygonal faces.

1. INTRODUCTION

For centuries the discussion on the idea of the four-colour theorem (4CT) caused controversy between scientists and mathematicians. The aim of this study is to investigate colouring of three-dimensional surfaces. Firstly, the concept and history of the 4CT is introduced. Differentiations and analysis between two-dimensional maps and three-dimensional surfaces are then presented. Finally, illustrative material will help to address the minimum number of colours needed to colour different polytopic surfaces and the implications of this to the field of the design will be discussed.

The famous 4CT is that a given map can be coloured with no more than four colours on its regions given the constraint that no two adjacent regions can be coloured by the same colour. The idea was first stated as far back as 1852 by Guthrie who realised that it is enough for most maps to be four-colourable and tried to find if that would be true for all maps; he later coined the term *four-colour conjecture* (4CC). H also discovered that there is a relation between vertices and edges such that each edge in map is incident with two vertices called a loop. Although he was the first to discover the theorem, he could not prove its existence; the 4CT was not accepted for another 100 years.

The first proof was attempted manually but the conjecture remained open and was not successfully proven until 1976 when Appel and Haken broke new ground by using computer programs to analyse more than 10,000 cases which took around 1200 hours to be analysed (Claude 2000). Unfortunately it was not fully accepted largely because the program they used was difficult to relate precisely to the formal statement of the mathematical theorem; the computer proof has not been corroborated by a more traditional approach.

2. METHOD

In order to analyse 2-D and 3-D patterns it is important to define all the concepts involved in this paper. Firstly, *M* for map, *V* for vertices, *E* means edges and *F* considering faces, and *G* means graph. Any plane with contiguous regions presents a map and the 4CT states that no more than four colours are required to colour each region under the condition that no two adjacent regions have the same colour. Like-coloured regions can share a point vertex but not an edge. The conjecture c made by Guthrie can be stated that c: v(G) → {0, 1, 2, 3} which means that for every edge of *G* with ends of *u* and *v*, c(u) ≠ c(v) (Bleecker 1996).

Regular polyhedra or polytopes can be generated in any number of dimensions. A polygon is an example of a 2-D polytope whereas a polyhedron polytope is 3-D type. In this paper a proof of the 4CT in the 3-D case is considered from a design perspectove with more complete independent hand-checking, which could be easier and clear to understand all readers. As our focuses is on the artistic, imaginative and creative aspect rather than mathematicals, we deformed the regular convex polyhedron with more *E* and *V* than the original, but have the same geometry and same angle. The portions of *F* increased lead also to a net increase. Therefore, for a surface *a* it required *n* different colours at the condition that no two adjacent *E* has the same colour. For example, the dodecahedron in Figure 1 we added one diagonal *E* across each *F*, connecting existing *V* and, particularly, it has the same faces and has twenty vertices. Whereas, for the tetrahedron we add *V* along some *E* in some *F*. Figures 1 and 2 illustrate a 3-D model of the dodecahedron and the tetrahedron and the unfolded net with different subdivisions.

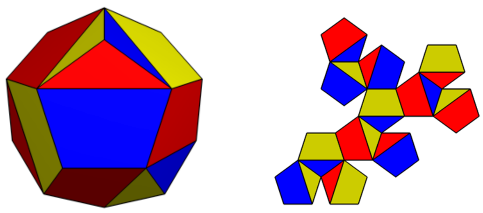

Figure 1: Three-colourable dodecahedron.

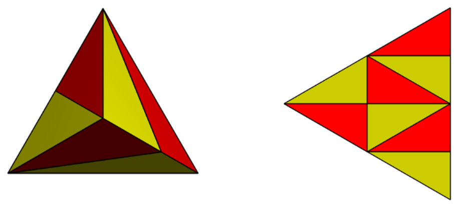

Figure 2: Two-colourable tetrahedron.

Next, Figures 3, 4 and 5 present the cube, icosahedron and the octahedron as 3-D models with each unfolded net. All the three shapes have no changes in its *F, V* or *E* number, and can be coloured with three or less colours.

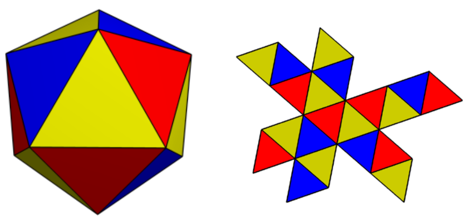

Figure 3: Three-colourable icosahedrons.

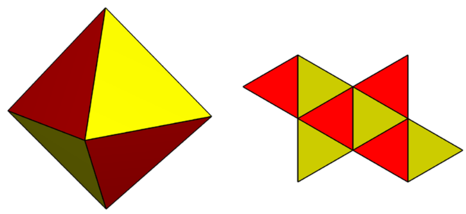

Figure 4: Two-colourable octahedron.

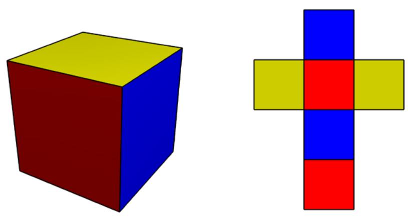

Figure 5: Three-colourable cube.

3. RESULTS AND DISCUSSION

Most work that has addressed extending 4CT to 3-D has considered solids (rather than surfaces) and the question of whether they share an *F* (rather than an *E*). Of course, for the 2-D or 3-D surface, it can be always subdivide such that each *N*-sided face becomes N+1 faces by connecting midpoints of subsequent *E*, creating *N* triangles around the *E*, plus one new *N*-gon in the middle (in this case *N* is the new vertices); then, we can colour all the triangles the same as the original face, and give the inner face any other colour thus maintaining the same number of colours as we aimed. Also, if the face is not a triangle, it divides the *N*-gon into *N* triangles with an extra vertex at the face's centre. Then each edge is divided into *L* segments, where *L* is the subdivision level chosen and the new *V* connected with new *E*. From view point of this study, It seems that the possibilities of colouring regular polyhedral with increasing number of polygonal *F, E* or *V* seems to be easier when having symmetries background, especially that the same manner could be extended from the 2-D planned (regular repeating) design in to 3-D surfaces.

Finally, Figure 6 illustrates an example of how such an extension could apply to the field of design. The example considered the developments beyond the 2-D plane to a 3-D surface by wrapping net to create the complete design of the dodecahedron. Mostly, just two colours are required within the whole shape, but a 3rd one is needed to counterchange between colours. Figure 6 has a dodecahedron with three subdivisions along each *E* coloured with no more than three colours under the condition that no two adjacent *E* have the same colour.

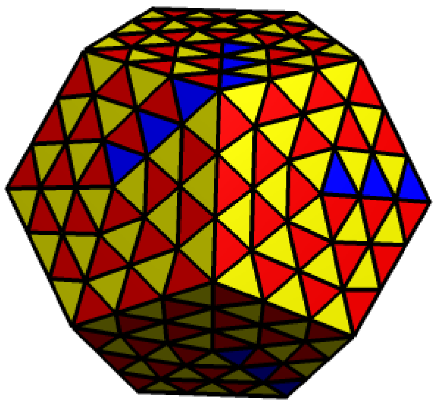

Figure 6: Three-colourable dodecahedron.

4. CONCLUSIONS

Undoubtedly, the significant help of computers lead Applel and Haken to prove the four-colour theorem. Nevertheless, computer programmes are known to be error-prone, and it is not easy to check the theorem using formal mathematics. Commencing this point, it might said that maybe because of the complex strategy and mathematical approached which has been used so far trying to solve and understand this theorem, it might be the case that needs a simple strategy and creative sense with artistic intuition involving artiest or designers. Further, the systematic colouring of 2-D repeating design resulted in limited number of colour combinations, and the application of two colours in conjunction with one or two fundamental symmetry, which appeared on the polyhedron faces. Also, considering the 3-D surface is a similar manner as observed in 2-D. The problem appears of a solid shapes rather than applying colour to the shapes faces which seems to works on certain 3-D shapes. Moreover the extension proved that it is possible for the regular polyhedral shapes to be three colourable surfaces with consideration of the basic of the 4CT.

REFERENCES

Claude, A.S., 2000. The Journey of the Four Colour Theorem Through Time, New Zealand *Mathematics Magazine* 38(3) 27-35.

Bleecker, D.B., 1996. Volume Increasing Isometric Deformations of Convex Polyhedra. *J. Differential Geometry* 43 505-526.

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